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aka SpongFest

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An Internal Model Principle for Synchronization in Heterogeneous Multi-Agent Systems

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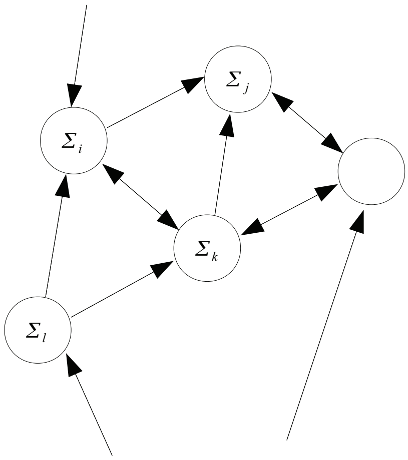
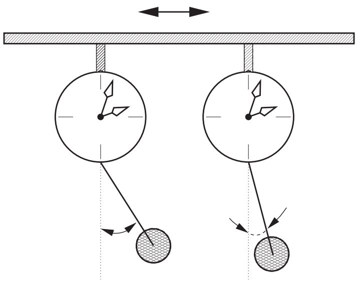
Consensus / Synchronization

Consensus / Synchronization

- deals with agreement about some common behavior in a group
- is relevant for all types of multi-agent systems

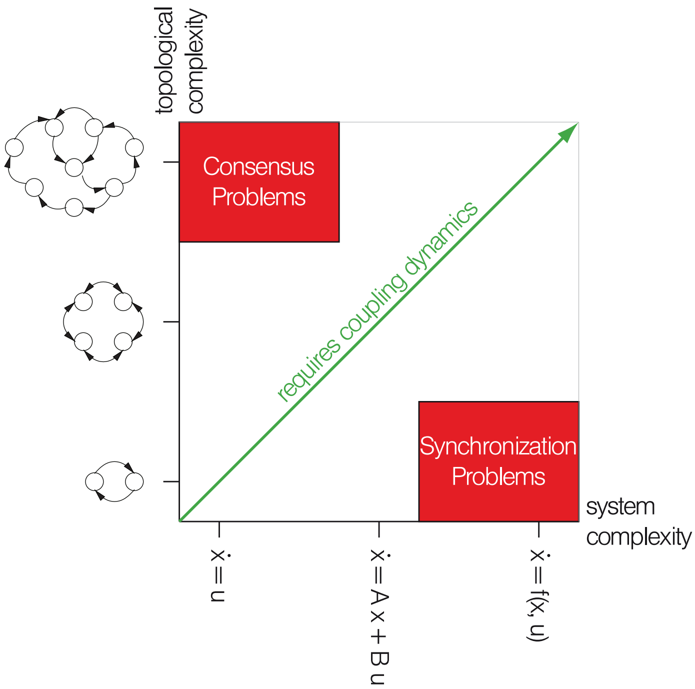




Output consensus/synchronization:

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$$

Consensus / Synchronization

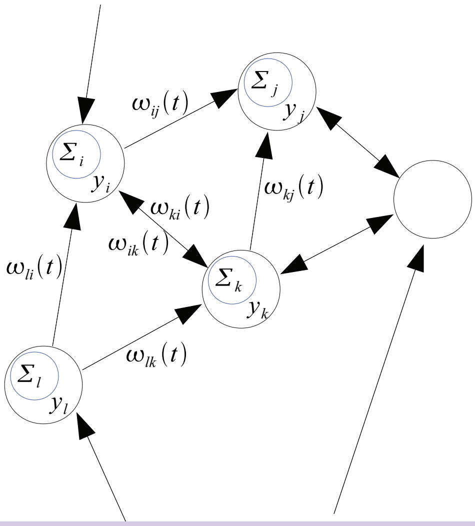


Typical problem setups:

- Consensus:
  simple systems,
  complex topologies.
- Synchronization:
  complex systems,
  simple topologies.

Extend to problems with high topological and system complexity!

General Problem Setup



- Each agent is a **nonlinear** dynamical system

$$\dot{x}_k(t) = f_k(x_k(t), u_k(t))$$
$$y_k(t) = h_k(x_k(t))$$

- Directed, time-varying graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), W(t)\}$, **uniformly connected** topology
- **nonidentical dynamics** of individual agents $\Rightarrow$ heterogeneous MAS

Very General Problem Setup

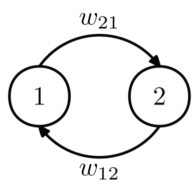
Output synchronization of heterogeneous, nonlinear MAS

Goal of the talk

1. Present necessary conditions for asymptotic synchronization
2. Show a synchronization procedure for classes of problems
3. Discuss cases in which exact synchronization is not possible

Example: Diffusively coupled scalar systems

$$\dot{y}_1(t) = -y_1(t) + u_1(t)$$
$$\dot{y}_2(t) = y_2(t) + u_2(t)$$



- **Diffusive couplings**:

$$u_1(t) = k_1 w_{12}(y_1(t) - y_2(t)),$$
$$u_2(t) = k_2 w_{21}(y_2(t) - y_1(t)).$$

- Find k_1, k_2 such that $(y_1 - y_2) \to 0$ as $t \to \infty$.
- **Observation**: Independently of w_{12}, w_{21}, encoding the interconnection topology, $(y_1 - y_2) \to 0$ as $t \to \infty$ if and only if $y_1 \to 0$ and $y_2 \to 0$ as $t \to \infty$ (e.g., $k_1 = 0$, $k_2 = -2/w_{21}$).

Only trivial synchronization is possible!

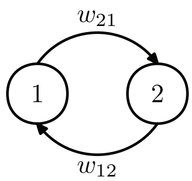
Example: Diffusively coupled linear systems

$$\dot{x}_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} u_1(t),$$

$$y_1(t) = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} x_1(t)$$

$$\dot{x}_2(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x_2(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2(t),$$

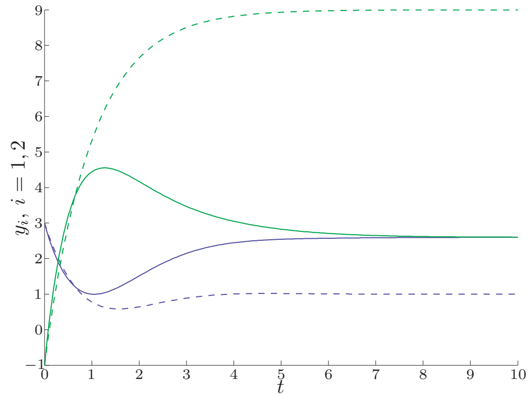
$$y_2(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x_2(t)$$



- **Diffusive couplings**:

$$u_1(t) = -w_{12}(y_1(t) - y_2(t)),$$
$$u_2(t) = -w_{21}(y_2(t) - y_1(t))$$

- **Observation**: Non-trivial synchronization occurs if $w_{12} \geq 0$, $w_{21} \geq 0$, and $\max(w_{12}, w_{21}) > 0$.



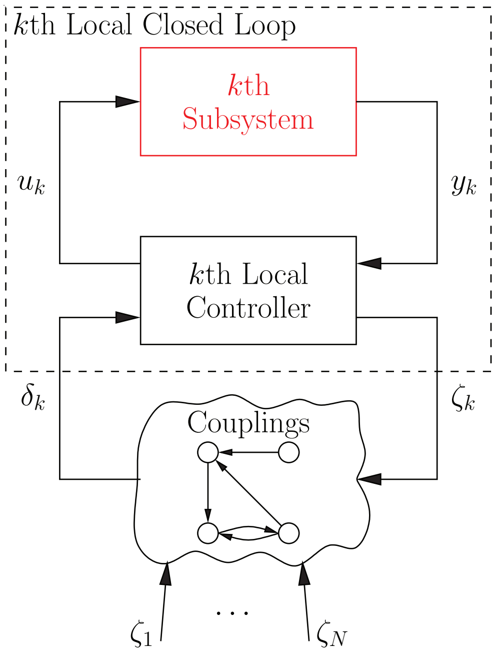
What are structural properties that allow synchronization?

Sync. of Non-Identical Systems – Problem Setup

Subsystems

$$\dot{x}_k(t) = A_k x_k(t) + B_k u_k(t)$$
$$y_k(t) = C_k x_k(t)$$

with state $x_k(t) \in \mathbb{R}^{n_k}$, input $u_k(t) \in \mathbb{R}^{p_k}$, and output $y_k(t) \in \mathbb{R}^q$.



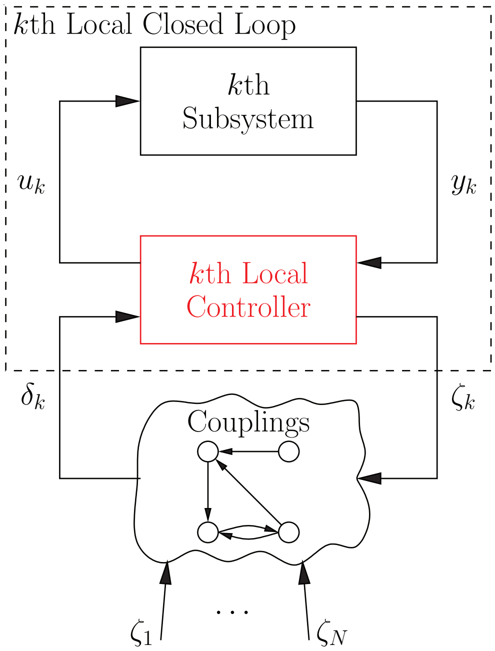
Sync. of Non-Identical Systems – Problem Setup

Subsystems

Local Controllers for Subsystems

$$
\begin{aligned}
\dot{z}_k(t) &= E_k z_k(t) + F_k \delta_k(t) + M_k y_k(t) \\
u_k(t) &= G_k z_k(t) + H_k \delta_k(t) + O_k y_k(t) \\
\zeta_k(t) &= P_k z_k(t) + Q y_k(t)
\end{aligned}
$$

with state $z_k(t) \in \mathbb{R}^{m_k}$, inputs $y_k(t) \in \mathbb{R}^q$ and $\delta_k(t) \in \mathbb{R}^r$, and outputs $\zeta_k(t) \in \mathbb{R}^r$ and $u_k(t) \in \mathbb{R}^{p_k}$.



Sync. of Non-Identical Systems – Problem Setup

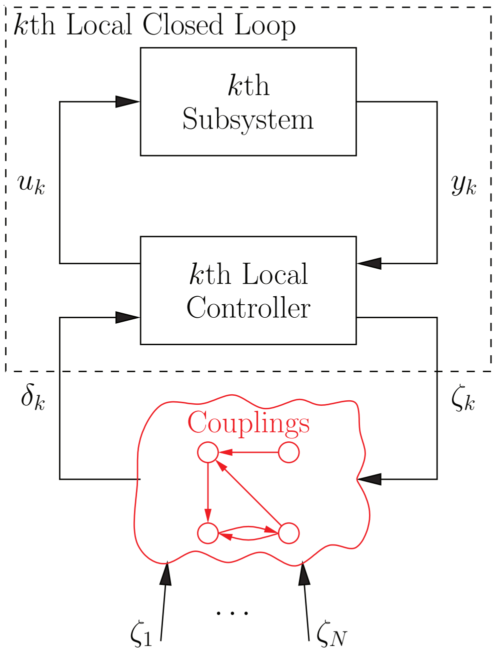
Subsystems

Local Controllers for Subsystems

Couplings

Diffusive couplings ($\simeq$ exchange of relative information)

$$
\delta_k(t) = \sum_{j=1}^{N} w_{kj}(t)\big(\zeta_k(t) - \zeta_j(t)\big)
$$



Sync. of Non-Identical Systems – Problem Setup

Subsystems

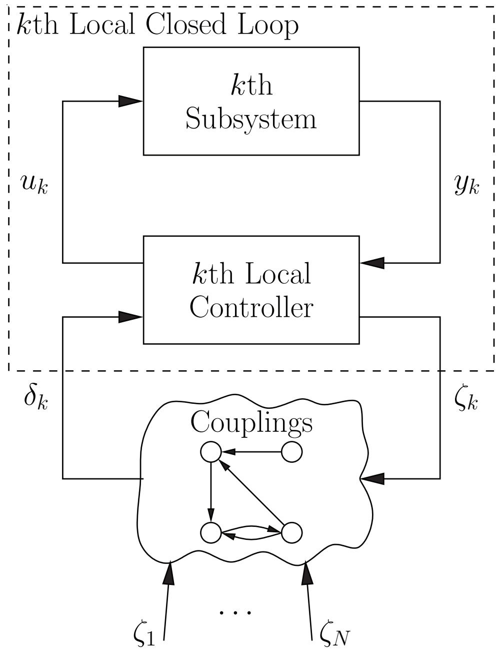
Local Controllers for Subsystems

Couplings

Control Objective

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0,$$
$$\lim_{t\to\infty} \|\zeta_i(t) - \zeta_j(t)\| = 0$$

exponentially fast for all i, j.



When do solutions exist?
How do solutions look like?
How do synchronous outputs look like?

Assumptions for Global Coupled System

Assumption: No trivial synchronization

- The global coupled system is not asymptotically stable.
- The global uncoupled system is detectable.

The long-term behavior is non-trivial and visible at the outputs.
Where do solutions live asymptotically?

Couplings use relative information, thus asymptotic synchronization implies $\delta_k(t) \to 0$, and therefore:

- The global coupled system is asymptotically autonomous.
- The limit system consists of decoupled local closed loop systems.

Synchronization occurs **asymptotically** if and only if there exists a **non-trivial attractive invariant subspace** such that restricted to this subspace the local closed loop systems are **decoupled** and **identically synchronous**.

Can we characterize the dynamics restricted to this invariant subspace?

Implicit Internal Model Principle

Local Closed Loop Systems

$$\left.\begin{aligned} \dot{x}_k^*(t) &= A^* x_k^*(t) + B_k^* \delta_k(t) \\ y_k(t) &= C_k^* x_k^*(t) \\ \zeta_k(t) &= P_k^* x_k^*(t) \end{aligned}\right\} \quad \text{subsystems + local controllers}$$

Theorem (Wieland and Allgöwer, 2009a)

If synchronization occurs asymptotically, then there exists a **virtual exosystem**

$$\dot{\xi}(t) = S\xi(t), \quad \eta(t) = R\xi(t) \tag{VEx}$$

with state $\xi(t) \in \mathbb{R}^\nu$ and output $\eta(t) \in \mathbb{R}^q$, and there exist matrices $\Psi_k \in \mathbb{R}^{(n_k+m_k)\times\nu}$ such that

$$\Psi_k S = A_k^* \Psi_k, \tag{Impl/a}$$

$$R = C_k^* \Psi_k. \tag{Impl/b}$$

In addition

$$\lim_{t\to\infty} (y_k(t) - \eta(t)) = 0$$

along some solution of (VEx).

Implicit Internal Model Principle

- Condition (Impl/a):

$$\Psi_k\dot{\xi}(t) = \Psi_k S\xi(t) = A_k^*\Psi_k\xi(t) = \dot{x}_k^*(t)\big|_{x_k^*(t)=\Psi_k\xi(t)}$$

holds for all $\xi(t) \in \mathbb{R}^\nu$.

"The subspace of $\mathbb{R}^\nu \times \mathbb{R}^{n_k+m_k}$ spanned by the columns of $\left(I_\nu, \Psi_k^T\right)^T$ is an invariant subspace for (VEx) + local closed loop system;
the dynamics restricted to this subspace is given by (VEx)."

- Condition (Impl/b):

$$\eta(t) = R\xi(t) = C_k^*\Psi_k\xi(t) = y_k(t)\big|_{x_k^*(t)=\Psi_k\xi(t)}$$

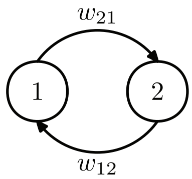
holds for all $\xi(t) \in \mathbb{R}^\nu$.

"When restricted to this subspace, $y_k(t) = \eta(t)$."

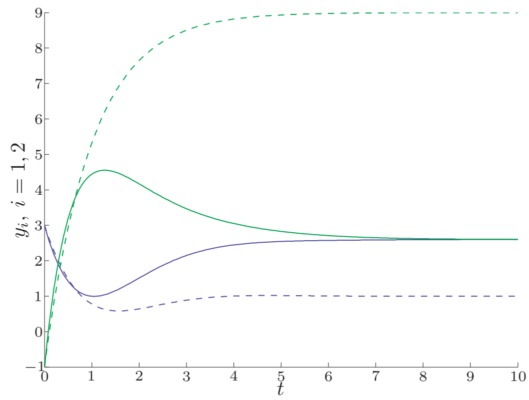
Synchronization implies that all local closed loops contain an **internal model** of a **common virtual exosystem**

Example: Diffusively coupled linear systems

$$\dot{x}_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} u_1(t),$$
$$y_1(t) = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} x_1(t)$$
$$\dot{x}_2(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x_2(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2(t),$$
$$y_2(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x_2(t)$$



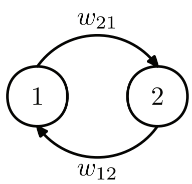
- **Virtual exosystem**:
$$\dot{\xi}(t) = 0,$$
$$\eta(t) = \xi(t)$$
- **Solution to** (VEx)**,** (Impl/a):
$$\Psi_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



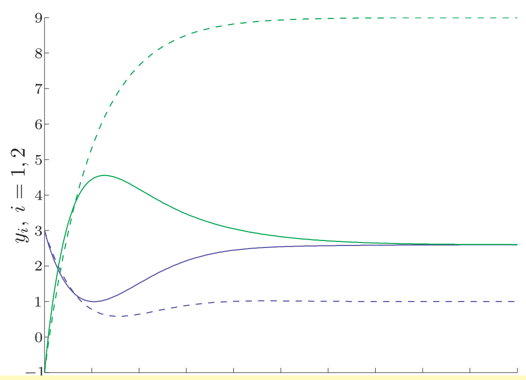
Both systems contain an internal model of an integrator and can therefore synchronize to solutions of an integrator.

Example: Diffusively coupled linear systems

$$\dot{x}_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} u_1(t),$$
$$y_1(t) = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} x_1(t)$$
$$\dot{x}_2(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x_2(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2(t),$$
$$y_2(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x_2(t)$$



- **Virtual exosystem**:
$$\dot{\xi}(t) = 0,$$
$$\eta(t) = \xi(t)$$
- **Solution to** (VEx)**,** (Impl/a):
$$\Psi_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Conditions (VEx), (Impl/a) are implicit in the sense that they depend on the local controllers, i.e., the solution to the problem.

Can we get rid of dependency on controllers?

Explicit Internal Model Principle

Theorem (Wieland and Allgöwer, 2009a)

If synchronization occurs asymptotically, then there exists a virtual exosystem (VEx) as before, and there exist matrices $\Pi_k \in \mathbb{R}^{n_k \times \nu}$, $\Lambda_k \in \mathbb{R}^{p_k \times \nu}$ such that

$$\Pi_k S = A_k \Pi_k + B_k \Lambda_k, \quad \text{(Expl/a)}$$

$$R = C_k \Pi_k. \quad \text{(Expl/b)}$$

- Condition (Expl/a) $\Rightarrow$ The subspace of $\mathbb{R}^\nu \times \mathbb{R}^{n_k}$ spanned by the columns of $\left(I_\nu, \Pi_k^T\right)^T$ is a controlled invariant subspace for (VEx) + subsystem,
  rendered invariant with the **feedforward** control $u_k(t) = \Lambda_k \xi(t)$
- Condition (Expl/b) is identical to condition (Impl/b)).
- Solvability of (Expl/a), (Expl/b) is equivalent to existence of a local controller that admits a solution of (Impl/a), (Impl/b).

Nonlinear Explicit Internal Model Principle

Subsystems:

$$\dot{x}_k(t) = f_k(x_k(t), u_k(t)), \quad y_k(t) = h_k(x_k(t))$$

with $x_k(t) \in \mathbb{R}^{n_k}$, $u_k(t) \in \mathbb{R}^{p_k}$, and $y_k(t) \in \mathbb{R}^q$.

Theorem (Wieland and Allgöwer, 2009b)

If the global coupled system is detectable and synchronization occurs asymptotically, then there exists a **virtual exosystem**

$$\dot{\xi}(t) = s(\xi(t)), \quad \eta(t) = \hat{h}(\xi(t)) \quad \text{(VEx)}$$

with state $\xi(t) \in \hat{\mathcal{X}}$ and output $\eta(t) \in \mathbb{R}^q$ characterizing the steady state dynamics, and there exist maps $\pi_k : \hat{\mathcal{X}} \to \mathbb{R}^{n_k}$, $\lambda_k : \hat{\mathcal{X}} \to \mathbb{R}^{p_k}$ such that

$$\frac{\partial \pi_k(\xi)}{\partial \xi} s(\xi) = f_k\big(\pi_k(\xi), \lambda_k(\xi)\big) \quad \text{(Expl/a)}$$

$$\hat{h}(\xi) = h_k(\pi_k(\xi)) \quad \text{(Expl/b)}$$

Conditions admit similar interpretation as in the linear case.
(Expl/a) $\Rightarrow$ invariance condition,
(Expl/b) $\Rightarrow$ subsystem output = virtual. exosystem output

Discussion of Internal Model Principle

Synchronization vs. Output Regulation

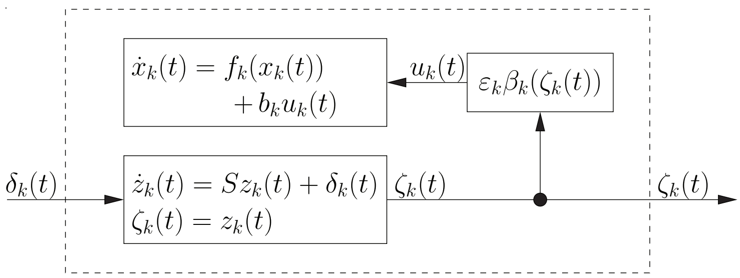
- Conditions (Expl/a), (Expl/b) correspond to the Francis-Equations, that are solvability conditions for the linear output regulation problem. (Nonlinear analogues exist!)
- Synchronization is not Output Regulation!
  - Output Regulation: the exosystem is an autonomous system external to the system to be controlled;
  - Synchronization: no autonomous exosystem exists, the virtual exosystem only exists internal to the network.

Synchronization of non-identical systems requires

- **Feedforward control** that ensures existence of an invariant set on which the network is identically synchronous
- **Feedback control** that renders this set attractive

The internal model conditions are **existence conditions** for the **feedforward** part of the **control**.

Synchronization of Non-Identical Oscillators



- Basic idea: synchronize copies of virtual exosystem ($\simeq$ coupling dynamics) and use synchronized signals to entrain oscillators.
- Coupling dynamics used to compensate for non-identical dynamics and to compensate for high topological complexity

Generic method to synchronize non-identical oscillators with weak assumptions on subsystems and couplings ($\simeq$ high system and topological complexity).

Example

Subsystems

Different Van der Pol oscillators (varying in parameter μ_k):

$$\dot{x}_k(t) = \begin{pmatrix} x_{k,2}(t) + \mu_k \left(x_{k,1}(t) - \frac{1}{3} x_{k,1}^3(t) \right) \\ -x_{k,1}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k(t)$$

k	1	2	3	4	5
μ_k	3.0	3.5	4.0	4.5	5.0
ω_k	0.7092	6.599	0.6158	0.5764	0.5411
T_k	8.859	9.521	10.20	10.90	11.61
ε_k	0.5	0.5	0.5	0.5	0.5
$\hat{\xi}_k$	4.065	4.578	4.960	5.310	5.704

Parameter values for simulation.

Couplings

Graph contains exactly one link at each time instant and switches every $T = 2.5$ units of time (seconds).

Example: Synchronization of Non-Identical Oscillators

figs/vdposcsync.mov

Questions Answered

1. *When do local controllers exist?*

   A **necessary condition** is solvability of the **explicit internal model equations** for some virtual exosystem.

2. *What are structural properties of the local controllers?*

   They solve the **implicit internal model equations**. Thus they contain a feedforward part that renders appropriate sets invariant with dynamics corresponding to the virtual exosystem dynamics.

3. *What are the dynamics of the synchronous outputs?*

   All possible synchronous outputs are given by outputs generated by the **virtual exosystem**

Further Questions

1. *What happens if a MAS does not fulfill the necessary condition?*
2. *Can we still achieve approximate/practical synchronization?*

Towards Practical Synchronization

Reformulation of the internal model principle for **static couplings**:

$$u_k(t) = K_k \sum_{j=1}^{N} a_{kj}(y_j(t) - y_k(t))$$

There exist matrices Π_k with full col rank, S and R, s.t. for $k = 1, \ldots, N$,

$$A_k \Pi_k = \Pi_k S, \tag{6}$$

$$C_k \Pi_k = R. \tag{7}$$

Two example networks, which do not fulfill conditions (6), (7)**:**

Harmonic oscillators

$$\dot{x}_k = \begin{bmatrix} 0 & \omega + \delta_k \\ -\omega - \delta_k & 0 \end{bmatrix} x_k + u_k,$$

$$y_k = x_k.$$

- Eq. (6) cannot be satisfied ✘

Double integrators

$$\dot{x}_k = \begin{bmatrix} 0 & 1 + \delta_k \\ 0 & 0 \end{bmatrix} x_k + u_k,$$

$$y_k = x_k.$$

- Eq. (7) cannot be satisfied ✘

Towards Practical Synchronization

Reformulation of the internal model principle for **static couplings**:

$$u_k(t) = K_k \sum_{j=1}^{N} a_{kj}(y_j(t) - y_k(t))$$

There exist matrices Π_k with full col rank, S and R, s.t. for $k = 1, \ldots, N$,

$$A_k \Pi_k = \Pi_k S, \tag{6}$$

$$C_k \Pi_k = R. \tag{7}$$

Two example networks, which do not fulfill conditions (6), (7)**:**

Harmonic oscillators

Double integrators

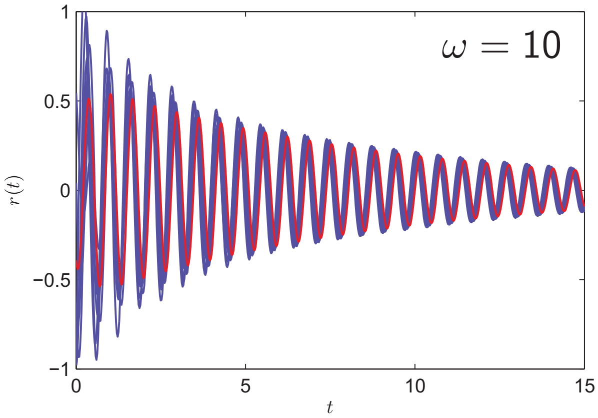
In both examples, exact synchronization is impossible.
What is the dynamic behavior of these networks?
Can we achieve practical synchronization?

- Eq. (6) cannot be satisfied ✘
- Eq. (7) cannot be satisfied ✘

Towards Practical Synchronization

Harmonic oscillators

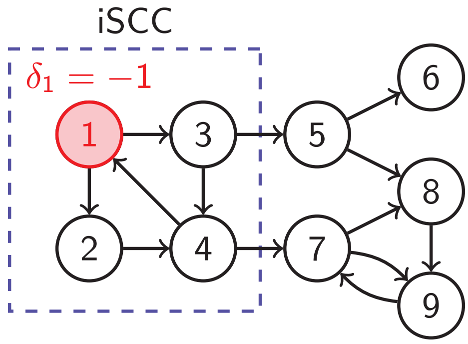
$$\dot{x}_k = \begin{bmatrix} 0 & \omega + \delta_k \\ -\omega - \delta_k & 0 \end{bmatrix} x_k + u_k,$$
$$y_k = x_k.$$



- Perturbation $\rightarrow$ asy. stability!
- Trivial synchronization!

Theorem (Seyboth et al., 2012)

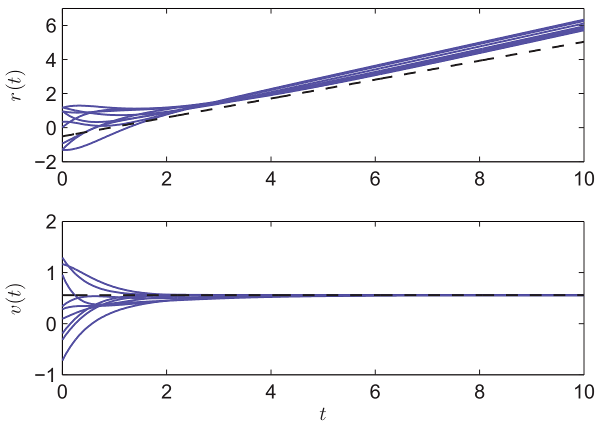
Suppose that the directed graph $\mathcal{G}$ is connected. Then, the network is asymptotically stable if and only if there exists a pair k, j of oscillators in the **iSCC** *of $\mathcal{G}$ such that $\delta_k \neq \delta_j$.*



Towards Practical Synchronization

Double integrators

$$\dot{x}_k = \begin{bmatrix} 0 & 1 + \delta_k \\ 0 & 0 \end{bmatrix} x_k + u_k,$$
$$y_k = x_k.$$



- $\lim_{t\to\infty} \|y_j(t) - y_k(t)\| \leq \epsilon$
- "Practical" synchronization!

Theorem (Seyboth et al., 2012)

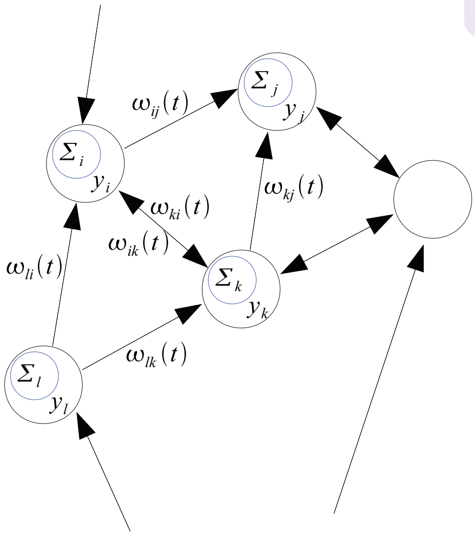
Suppose that the directed graph $\mathcal{G}$ is connected and there exists a pair k, j of agents such that $\delta_k \neq \delta_j$. Then, the second states $v(t)$ synchronize and the first states $r(t)$ reach constant offsets $r_\perp$, given by

$$\begin{bmatrix} r_\perp \\ c \end{bmatrix} = \begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} \delta p^\top v_0 \\ 0 \end{bmatrix}.$$

where $c \in \mathbb{R}$, $r_\perp \in \mathbb{R}^N$, $\mathbf{1}^\top r_\perp = 0$.

Scaling the weights in $\mathcal{G}$ by a gain $\gamma > 1$ scales $\|r_\perp\|$ by $1/\gamma < 1$.

Summary



Key result: **Internal Model Principle for Synchronization**

- Presents a necessary condition for output synchronization.
- Links synchronization problems to output regulation problems.
- Suggests a control paradigm for output synchronization of heterogenous MAS using dynamic couplings.
- Presented a new result for synchronization of nonlinear oscillators over uniformly connected communication graphs.

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Happy Birthday Mark !!!

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